

ON THE CHARACTER OF CERTAIN IRREDUCIBLE MODULAR REPRESENTATIONS

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1. Let G be an almost simple, simply connected algebraic group over \mathbf{k} , an algebraically closed field of characteristic $p > 1$. Let $\text{Rep}G$ be the category of finite dimensional \mathbf{k} -vector spaces with a given rational linear action of G and let $\text{Irr}G$ be a set of representatives for the simple objects of $\text{Rep}G$. We fix a Borel subgroup B of G and a maximal torus T of B ; let $Y = \text{Hom}(\mathbf{k}^*, T)$, $X = \text{Hom}(T, \mathbf{k}^*)$ (with group operation written as $+$) and let $\langle, \rangle : Y \times X \rightarrow \mathbf{Z}$ be the obvious pairing. If $V \in \text{Irr}G$ then there is a well defined $\lambda_V \in X$ with the following property: the T -action on the unique B -stable line in V is through λ_V ; according to Chevalley, $E \mapsto \lambda_E$ is a bijection from $\text{Irr}G$ to a subset of X of the form

$$X^+ = \{\lambda \in X; \langle \check{\alpha}_i, \lambda \rangle \in \mathbf{N} \quad \forall i \in I\}$$

for a well defined basis $\{\check{\alpha}_i; i \in I\}$ of Y . For $\lambda \in X^+$ we shall denote by V_λ the object of $\text{Irr}G$ corresponding to λ . If $V \in \text{Rep}G$, then for any $\mu \in X$ we denote by $n_\mu(V)$ the multiplicity of μ in $V|_T$; we set $[V] = \sum_{\mu \in X} n_\mu(V) e^\mu \in \mathbf{Z}[X]$ where $\mathbf{Z}[X]$ is the group ring of X (the basis element of $\mathbf{Z}[X]$ corresponding to $\mu \in X$ is denoted by e^μ so that $e^\mu e^{\mu'} = e^{\mu+\mu'}$ for $\mu, \mu' \in X$). It is of considerable interest to compute explicitly the element $[V_\lambda] \in \mathbf{Z}[X]$ for any $\lambda \in X^+$. Let h be the Coxeter number of G . A conjectural formula for $[V_\lambda]$ (assuming that $p \geq c_G^0$ where c_G^0 is a constant depending only on the root datum of G) was stated in [L1, p.316]. In the early 1990's it was proved (see [AJS] and the references there) that there exists a (necessarily unique) prime number $c_G \geq c_G^0$ depending only on the root datum of G such that the conjectural formula in [L1, p.316] is true if $p \geq c_G$ and c_G is minimum possible (but c_G was not explicitly determined). In [Fi], Fiebig showed that $c_G \leq c'_G$ where c'_G is an explicitly known but very large constant. In [Wi], Williamson, partly in collaboration with Xuhua He, showed that for infinitely many G , c_G is much larger than c_G^0 . Now the conjecture in [L1, p.316] had an unsatisfactory aspect: it applied only to a finite set of $\lambda \in X^+$ which, after application of Jantzen's results [Ja] on translation functors, becomes a larger

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but still finite set (including all λ in $X_{red}^+ = \{\lambda \in X^+; \langle \check{\alpha}_i, \lambda \rangle \leq p-1 \ \forall i \in I\}$); then the case of a general $\lambda \in X^+$ had to be obtained by applying the Steinberg tensor product theorem [St]. In this note I want to offer a reformulation of the conjecture in [L1, p.316] (now a known theorem for p large enough) which applies directly to any $\lambda \in X^+$, see 7(b).

2. Notation. Let NT be the normalizer of T in G and let $W = NT/T$ be the Weyl group. Note that W acts naturally on T hence on Y, X and $\mathbf{Z}[X]$. Let $\check{R} = \{y \in Y; y = w(\check{\alpha}_i) \text{ for some } w \in W, i \in I\}$ (the set of coroots). Define $\check{\alpha}_0 \in \check{R}$ by the condition that $\check{\alpha}_0 + \check{\alpha}_i \notin \check{R}$ for any $i \in I$. (Thus $\check{\alpha}_0$ is the highest coroot). For $i \in I \sqcup \{0\}$ define $\alpha_i \in X$ by the condition that the map $X \rightarrow X, \lambda \mapsto \lambda - \langle \check{\alpha}_i, \lambda \rangle \alpha_i$, is induced by a (uniquely defined) element s_i of W . Let $w \mapsto \epsilon_w$ be the homomorphism $W \rightarrow \{1, -1\}$ such that $\epsilon_{s_i} = -1$ for any $i \in I$. Define $\rho \in X^+$ by $\langle \check{\alpha}_i, \rho \rangle = 1$ for any $i \in I$. Let \leq be the partial order on X given by $\lambda \leq \lambda'$ whenever $\lambda' - \lambda \in \sum_{i \in I} \mathbf{N} \alpha_i$.

3. Let

$$\Delta = \{\lambda \in X; \langle \check{\alpha}_i, \lambda + \rho \rangle \leq 0 \ \forall i \in I, \langle \check{\alpha}_0, \lambda + \rho \rangle \geq -p\}.$$

For $i \in I$ we define $s'_i : X \rightarrow X$ by $s'_i(\lambda) = \lambda - \langle \check{\alpha}_i, \lambda + \rho \rangle \alpha_i$ (an affine reflection). We define $s'_0 : X \rightarrow X$ by $s'_0(\lambda) = \lambda - (\langle \check{\alpha}_0, \lambda + \rho \rangle + p) \alpha_0$ (an affine reflection). Let W_a be the subgroup of the group of permutations of X generated by $s'_i (i \in I \cup \{0\})$. Then W_a is a Coxeter group on the generators $s'_i (i \in I \cup \{0\})$, with length function $l : W_a \rightarrow \mathbf{N}$.

For $\lambda \in X$ we have $w^{-1}(\lambda) \in \Delta$ for some $w \in W_a$ and among all such w there is a unique one, w_λ , of minimal length.

For $\lambda, \mu \in X^+$ we set $w = w_\lambda$ and

$$\mathbf{p}_{\mu, \lambda} = \sum_{y \in W_a; y^{-1}(\mu) = w^{-1}(\lambda)} (-1)^{l(yw)} P_{y, w}(1) \in \mathbf{Z}$$

where $P_{y, w}$ is the polynomial associated in [KL] to y, w in the Coxeter group W_a .

From the definitions we see that $\mathbf{p}_{\mu, \lambda} \neq 0 \implies \mu \leq \lambda, \mathbf{p}_{\lambda, \lambda} = 1$. Hence for $\lambda, \mu \in X^+$ we can define $\mathbf{q}_{\mu, \lambda} \in \mathbf{Z}$ by the requirement

$$\sum_{\nu \in X^+} \mathbf{p}_{\mu, \nu} \mathbf{q}_{\nu, \lambda} = \delta_{\mu, \lambda}$$

for any λ, μ in X^+ . We have $\mathbf{q}_{\mu, \lambda} \neq 0 \implies \mu \leq \lambda, \mathbf{q}_{\lambda, \lambda} = 1$.

4. For any $\lambda \in X^+$ we can write uniquely $\lambda = \sum_{k \geq 0} p^k \lambda^k$ where $\lambda^k \in X_{red}^+$ for all $k \geq 0$ and $\lambda^k = 0$ for large k .

For any $\lambda \in X^+$ and any $k \in \mathbf{N}$ we define elements $E_\lambda^k \in \mathbf{Z}[X]$ by induction on k as follows:

$$E_\lambda^0 = \sum_{w \in W} \epsilon_w e^{w(\lambda + \rho)} / \sum_{w \in W} \epsilon_w e^{w(\rho)},$$

$$(a) \quad E_{\lambda}^k = \sum_{\mu \in X^+} \mathbf{p}_{\mu, \sum_{j; j \geq k-1} p^{j-k+1} \lambda^j} E_{\sum_{j; 0 \leq j \leq k-2} p^j \lambda^j + p^{k-1} \mu}^{k-1} \text{ if } k \geq 1.$$

Note that $E_{\lambda}^k \in \mathbf{Z}[X]^W$, the ring of W -invariants in $\mathbf{Z}[X]$.

We show that for $k \geq 0$ we have

$$(b) \quad E_{\lambda}^k = \sum_{\mu \in X^+} \mathbf{q}_{\mu, \sum_{j; j \geq k} p^{j-k} \lambda^j} E_{\sum_{j; 0 \leq j \leq k-1} p^j \lambda^j + p^k \mu}^{k+1}.$$

For any $\mu \in X^+$ and any $k, h \geq 0$,

$$\left(\sum_{j; 0 \leq j \leq k-1} p^j \lambda^j + p^k \mu \right)^h$$

is equal to λ^h if $0 \leq h \leq k-1$ and to μ^{h-k} if $h \geq k$; hence, by (a), we have

$$E_{\sum_{j; 0 \leq j \leq k-1} p^j \lambda^j + p^k \mu}^{k+1} = \sum_{\nu \in X^+} \mathbf{p}_{\nu, \sum_{h; h \geq k} p^{h-k} \mu^{h-k}} E_{\sum_{h; 0 \leq h \leq k-1} p^h \lambda^h + p^k \nu}^k.$$

Thus the right hand side of (b) is

$$\begin{aligned} & \sum_{\mu \in X^+} \mathbf{q}_{\mu, \sum_{j; j \geq k} p^{j-k} \lambda^j} \sum_{\nu \in X^+} \mathbf{p}_{\nu, \sum_{h; h \geq k} p^{h-k} \mu^{h-k}} E_{\sum_{h; 0 \leq h \leq k-1} p^h \lambda^h + p^k \nu}^k \\ &= \sum_{\mu \in X^+} \mathbf{q}_{\mu, \sum_{j; j \geq k} p^{j-k} \lambda^j} \sum_{\nu \in X^+} \mathbf{p}_{\nu, \mu} E_{\sum_{h; 0 \leq h \leq k-1} p^h \lambda^h + p^k \nu}^k \\ &= \sum_{\nu \in X^+} \delta_{\nu, \sum_{j; j \geq k} p^{j-k} \lambda^j} E_{\sum_{h; 0 \leq h \leq k-1} p^h \lambda^h + p^k \nu}^k \\ &= E_{\sum_{h; 0 \leq h \leq k-1} p^h \lambda^h + p^k \sum_{j; j \geq k} p^{j-k} \lambda^j}^k \\ &= E_{\lambda}^k. \end{aligned}$$

This proves (b).

By induction on k we see, using (a), that for any $k \geq 1$ we have

$$\begin{aligned} E_{\lambda}^k &= \sum_{\mu_0, \mu_1, \dots, \mu_{k-1} \text{ in } X^+} \mathbf{p}_{\mu_0, \lambda^0 + p\mu_1} \mathbf{p}_{\mu_1, \lambda^1 + p\mu_2} \cdots \mathbf{p}_{\mu_{k-2}, \lambda^{k-2} + p\mu_{k-1}} \\ (c) \quad & \times \mathbf{p}_{\mu_{k-1}, \sum_{j \geq k-1} p^{j-k+1} \lambda^j} E_{\mu_0}^0. \end{aligned}$$

By induction on k we see, using (b), that for any $k \geq 1$ we have

$$\begin{aligned} E_{\lambda}^0 &= \sum_{\nu_0, \nu_1, \dots, \nu_{k-1} \text{ in } X^+} \mathbf{q}_{\nu_0, \lambda} \mathbf{q}_{\nu_1, (\nu_0 - \nu_0^0)/p} \mathbf{q}_{\nu_2, (\nu_1 - \nu_1^0)/p} \cdots \mathbf{q}_{\nu_{k-1}, (\nu_{k-2} - \nu_{k-2}^0)/p} \\ (d) \quad & \times E_{\nu_0^0 + p\nu_1^0 + \cdots + p^{k-2}\nu_{k-2}^0 + p^{k-1}\nu_{k-1}}^k. \end{aligned}$$

5. Let $\lambda \in X^+$. We can find $n \geq 0$ such that $\lambda^n = \lambda^{n+1} = \dots = 0$. If $k \geq n$ we have

$$E_\lambda^k = E_\lambda^{k+1}.$$

Indeed, if $\mu \in X^+$ and $\mathbf{q}_{\mu, \sum_{j; j \geq k} p^{j-k} \lambda^j} \neq 0$ we have $\mathbf{q}_{\mu, 0} \neq 0$ hence $\mu \leq 0$, $\mu = 0$ and $\mathbf{q}_{0,0} = 1$; it follows that

$$E_\lambda^k = E_{\sum_{j; 0 \leq j \leq k-1} p^j \lambda^j}^{k+1} = E_\lambda^{k+1}.$$

Thus we can set $E_\lambda^\infty = E_\lambda^k$ for large k . Clearly, if $\lambda \in X_{red}^+$, then $E_\lambda^1 = E_\lambda^2 = \dots = E_\lambda^\infty$.

Letting $k \rightarrow \infty$ in 4(c),(d), we deduce that

$$E_\lambda^\infty = \sum_{\mu_0, \mu_1, \mu_2, \dots \text{ in } X^+; \mu_h=0 \text{ for large } h} (\mathbf{p}_{\mu_0, \lambda^0 + p\mu_1} \mathbf{p}_{\mu_1, \lambda^1 + p\mu_2} \mathbf{p}_{\mu_2, \lambda^2 + p\mu_3} \dots) E_{\mu_0}^0,$$

(note that for large h we have $\mathbf{p}_{\mu_h, \lambda^h + p\mu_{h+1}} = \mathbf{p}_{0,0} = 1$ so that the infinite product makes sense) and

$$E_\lambda^0 = \sum_{\nu_0, \nu_1, \nu_2, \dots \text{ in } X^+} (\mathbf{q}_{\nu_0, \lambda} \mathbf{q}_{\nu_1, (\nu_0 - \nu_0^0)/p} \mathbf{q}_{\nu_2, (\nu_1 - \nu_1^0)/p} \dots) E_{\nu_0^0 + p\nu_1^0 + p^2\nu_2^0 + \dots}^\infty$$

(note that for large h we have $\nu_h = 0$ hence $\mathbf{q}_{\nu_{h+1}, (\nu_h - \nu_h^0)/p} = \mathbf{q}_{0,0} = 1$ so that the infinite product makes sense).

6. It is known since the early 1990's that, if p is not very small, then the conjecture 8.2 in [L2] on quantum groups at a p -th root of 1 holds. In particular for $\lambda \in X^+$, the element E_λ^1 describes the character of an irreducible finite dimensional representation of such a quantum group and the tensor product theorem [L2, 7.4] holds for it.

Thus, if for any $\xi = \sum_{\lambda \in X} c_\lambda e^\lambda \in \mathbf{Z}[X]$ (with $c_\lambda \in \mathbf{Z}$) and any $h \geq 0$ we set $\xi^{(h)} = \sum_{\lambda \in X} c_\lambda e^{p^h \lambda} \in \mathbf{Z}[X]$, then for any $\lambda \in X^+$ we have the equality

$$(a) \quad E_\lambda^1 = E_{\lambda^0}^1 (E_{\sum_{j \geq 1} p^{j-1} \lambda^j}^0)^{(1)}.$$

We show by induction on $k \geq 1$ that for any $\lambda \in X^+$ we have

$$(b) \quad E_\lambda^k = E_{\lambda^0}^1 (E_{\lambda^1}^1)^{(1)} \dots (E_{\lambda^{k-1}}^1)^{(k-1)} (E_{\sum_{j \geq k} p^{j-k} \lambda^j}^0)^{(k)}.$$

By (a), we can assume that $k \geq 2$. Using 4(a), it is enough to show that

$$\begin{aligned} & \sum_{\mu \in X^+} \mathbf{p}_{\mu, \sum_{j; j \geq k-1} p^{j-k+1} \lambda^j} E_{\sum_{j; 0 \leq j \leq k-2} p^j \lambda^j + p^{k-1} \mu}^{k-1} \\ &= E_{\lambda^0}^1 (E_{\lambda^1}^1)^{(1)} \dots (E_{\lambda^{k-1}}^1)^{(k-1)} (E_{\sum_{j \geq k} p^{j-k} \lambda^j}^0)^{(k)}. \end{aligned}$$

Replacing here

$$E_{\sum_{j;0 \leq j \leq k-2} p^j \lambda^j + p^{k-1} \mu}^{k-1} = E_{\lambda^0}^1 (E_{\lambda^1}^1)^{(1)} \dots (E_{\lambda^{k-2}}^1)^{(k-2)} (E_{\sum_{j \geq k-1} p^{j-k+1} \mu^j}^0)^{(k-1)}$$

which is known from the induction hypothesis, we see that it is enough to show that

$$\begin{aligned} \sum_{\mu \in X^+} \mathbf{p}_{\mu, \sum_{j; j \geq k-1} p^{j-k+1} \lambda^j} E_{\lambda^0}^1 (E_{\lambda^1}^1)^{(1)} \dots (E_{\lambda^{k-2}}^1)^{(k-2)} (E_{\sum_{j \geq k-1} p^{j-k+1} \mu^j}^0)^{(k-1)} \\ = E_{\lambda^0}^1 (E_{\lambda^1}^1)^{(1)} \dots (E_{\lambda^{k-1}}^1)^{(k-1)} (E_{\sum_{j \geq k} p^{j-k} \lambda^j}^0)^{(k)}. \end{aligned}$$

Thus, it is enough to show that

$$\sum_{\mu \in X^+} \mathbf{p}_{\mu, \sum_{j; j \geq k-1} p^{j-k+1} \lambda^j} (E_{\sum_{j \geq k-1} p^{j-k+1} \mu^j}^0)^{(k-1)} = (E_{\lambda^{k-1}}^1)^{(k-1)} (E_{\sum_{j \geq k} p^{j-k} \lambda^j}^0)^{(k)} \quad \blacksquare$$

or that

$$\sum_{\mu \in X^+} \mathbf{p}_{\mu, \sum_{j; j \geq k-1} p^{j-k+1} \lambda^j} E_{\sum_{j \geq k-1} p^{j-k+1} \mu^j}^0 = E_{\lambda^{k-1}}^1 (E_{\sum_{j \geq k} p^{j-k} \lambda^j}^0)^{(1)}.$$

Using (a), the right hand side is $E_{\sum_{j \geq k-1} p^{j-k+1} \lambda^j}^1$. This is equal to the left hand side, by 4(a). This proves (b).

Letting $k \rightarrow \infty$ in (b) we obtain for any $\lambda \in X^+$:

$$(c) \quad E_{\lambda}^{\infty} = E_{\lambda^0}^1 (E_{\lambda^1}^1)^{(1)} (E_{\lambda^2}^1)^{(2)} \dots$$

Note that for large h we have $\lambda^h = 0$ hence $E_{\lambda^h}^1 = 1$ so that the infinite product makes sense. (We have also used that $E_{\sum_{j \geq k} p^{j-k} \lambda^j}^0 = E_0^0 = 1$ for large h .)

7. We now assume that $p \geq c_G$. Then, by the first paragraph of no.6, we have

$$(a) \quad [V_{\lambda}] = E_{\lambda}^1 \text{ for any } \lambda \in X_{red}^+.$$

Using the Steinberg tensor product theorem [St] and (a), we see that for any $\lambda \in X^+$ we have

$$[V_{\lambda}] = [V_{\lambda^0}] [V_{\lambda^1}]^{(1)} [V_{\lambda^2}]^{(2)} \dots = E_{\lambda^0}^1 (E_{\lambda^1}^1)^{(1)} (E_{\lambda^2}^1)^{(2)} \dots$$

Using this and 6(c) we deduce

$$(b) \quad [V_{\lambda}] = E_{\lambda}^{\infty}.$$

8. We preserve the setup of no.7. In the identity

$$E_\lambda^0 = \sum_{\mu \in X^+} \mathbf{q}_{\mu, \lambda} E_\mu^1$$

(see 4(b)) the coefficient $\mathbf{q}_{\mu, \lambda}$ can be interpreted as the multiplicity of an irreducible representation of a quantum group at a p -th root of 1 in a not necessarily irreducible representation of that quantum group. In particular we have

$$(a) \quad \mathbf{q}_{\mu, \lambda} \in \mathbf{N}$$

for any λ, μ in X^+ . We show by descending induction on k that for any $\lambda \in X^+$ and any $k \geq 0$ we have

$$(b) \quad E_\lambda^k = [V_\lambda(k)]$$

for some $V_\lambda(k) \in \text{Rep}G$.

If k is large, we have $E_\lambda^k = E_\lambda^\infty$ and (b) follows from 7(b). Assume now that $k \geq 0$ and that (b) is known when k is replaced by $k+1$. Then (b) holds for k by (a), 4(b) and the induction hypothesis.

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